Math 352 HW. # 4

Homework problems are taken from “Real Analysis” by N. L. Carothers. The problems are color coded to indicate level of difficulty. The color green indicates an elementary problem, which you should be able to solve effortlessly. Yellow means that the problem is somewhat harder. Red indicates that the problem is hard. You should attempt the hard problems especially.

Unless the contrary is explicitly stated, all numbers that are mentioned in these exercises are understood to be real.

1. Suppose that $f_n : [a,\ b] \to \mathbb{R}$ is an increasing function for each n, and that $f(x) = \lim_{n\to\infty} f_n(x)$ exists for each x in [*a*, *b*]. Is *f* increasing?

2. Let $f_n : [a,\ b] \to \mathbb{R}$ satisfy $|f_n(x)| \le 1$ for all x and n. Show that there is a sequence $\{f_{n_k}\}$ such that $\lim_{k\to\infty} f_{n_k}(x)$ exists for each *rational* x in [*a*, *b*]. [Hint: This is a “diagonalization” argument.]

3. Let $\{f_n\}$ and $\{g_n\}$ be real-valued functions on a set X, and suppose that $\{f_n\}$ and $\{g_n\}$ converge uniformly on X. Show that $\{f_n + g_n\}$ converges uniformly on X. Give an example showing that $\{f_n g_n\}$ need not converge uniformly on X (although it will converge pointwise, of course).

4. Let $f_n : \mathbb{R} \to \mathbb{R}$, and suppose that $f_n \rightrightarrows 0$ on every closed, bounded interval [*a*, *b*]. Does it follow that $f_n \rightrightarrows 0$ on $\mathbb{R}$? Explain.

5. Let $f : \mathbb{R} \to \mathbb{R}$ be uniformly continuous, and define $f_n(x) = f(x + (1/n))$. Show that $f_n \rightrightarrows f$ on $\mathbb{R}$.

6. Let (X, d) and (Y, p) be metric spaces, and let $f,\ f_n : X \to Y$ with $f_n \rightrightarrows f$ on X. If each f_n is continuous at $x \in X$, and if $x_n \to x$ in X, prove that $\lim_{n\to\infty} f_n(x_n) = f(x)$.

7. *Dini’s theorem*. Let X be a compact metric space, and suppose that the sequence $\{f_n\}$ in C(X) increases pointwise to a *continuous* function $f \in$ C(X); that is, $f_n(x) \le f_{n+1}(x)$ for each n and x, and $f_n(x) \to f(x)$ for each x. Prove that the convergence is actually uniform. The same is true if $\{f_n\}$ decreases pointwise to *f*. [Hint: First reduce to the case where $\{f_n\}$ decreases pointwise to 0. Now, given $\epsilon >$

0, consider the (open) sets $U_n = \{x \in X : f_n(x) < \varepsilon\}$. Give an example showing that $f \in C(X)$ is necessary.

8. Suppose that $\{f_n\}$ is a sequence of functions in C[0, 1] and that $f_n \overset{\rightarrow}{\rightarrow} f$ on [0, 1]. True or false? $\int_0^{1-(1/n)} f_n \to \int_0^1 f$.

9. Show that B(X) is an algebra of functions: that is, if *f*, *g* ϵ B(X), then so is *fg* and $\|fg\|_\infty \le \|f\|_\infty \|g\|_\infty$. Moreover, if $f_n \to f$ and $g_n \to g$ in B(X), show that $f_n g_n \to fg$ in B(X). That is, if $f_n \overset{\rightarrow}{\rightarrow} f$ and $g_n \overset{\rightarrow}{\rightarrow} g$, then $f_n g_n \overset{\rightarrow}{\rightarrow} fg$. Compare this with exercise 3.

10. Show that $\sum_{n=1}^{\infty} x^2/(1+x^2)^n$ converges for all $|x| \le 1$, but that the convergence is *not* uniform. [Hint: Find the sum!]

11. Let $f_n : \mathbb{R} \to \mathbb{R}$ be continuous, and suppose that $\{f_n\}$ converges uniformly on $\mathbb{Q}$. Show that $\{f_n\}$ actually converges uniformly on all of $\mathbb{R}$. [Hint: Show that $\{f_n\}$ is uniformly Cauchy.]

12. (a) For which values of x does $\sum_{n=1}^{\infty} ne^{-nx}$ converge? On which intervals is the convergence uniform?

(b) Conclude that $\int_1^2 \sum_{n=1}^{\infty} ne^{-nx}\,dx = e/(e^2 - 1)$.

13. Prove that $\sum_{n=1}^{\infty} x/[n^\alpha(1+nx^2)]$ converges uniformly on every bounded interval in $\mathbb{R}$ provided that $\alpha > 1/2$. Is the convergence uniform on all of $\mathbb{R}$?

14. Show that $\lim_{x\to 1} \sum_{n=1}^{\infty} nx^2/(n^3 + x^2) = \sum_{n=1}^{\infty} n/(n^3 + 1)$.

15. (a) If $\sum_{n=1}^{\infty} |a_n| < \infty$, show that $\sum_{n=1}^{\infty} a_n e^{-nx}$ is uniformly convergent on [0, ∞).

(b) If we assume only that $\{a_n\}$ is bounded, show that $\sum_{n=1}^{\infty} a_n e^{-nx}$ is uniformly convergent on [δ, ∞) for every $\delta > 0$.

16. Let $0 \le g_n \in C[a,\ b]$. If $\sum_{n=1}^{\infty} g_n$ converges pointwise to a continuous function on $[a, b]$, show that $\sum_{n=1}^{\infty} g_n$ converges uniformly on $[a, b]$.

17. Let $\{f_n\}$ be a sequence of continuous functions on $(0, \infty)$ with $|f_n(x)| \le n$ for every x > 0 and $n \ge 1$, and such that $\lim_{x\to\infty} f_n(x) = 0$ for each n. Show that $f(x) = \sum_{n=1}^{\infty} 2^{-n} f_n(x)$ defines a continuous function on $(0, \infty)$ that also satisfies $\lim_{x\to\infty} f(x) = 0$.